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Analysis of the structure of the boundary layer in the problem of the torsion of a laminated spherical shell☆

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ABSTRACT

The structures of the boundary layer in the problem of the torsion of a radially stratified spherical segment (shell) with an arbitrary number of alternating hard and soft layers are investigated. It is shown that weakly attenuating boundary-layer solutions exist. Despite the fact that a stress state, self-balanced in the section, corresponds to these elementary solutions, they may penetrate fairly deeply and considerably change the stress–strain state pattern far from the ends. Using an asymptotic analysis of the problem, an applied theory of torsion is proposed which takes into account weakly attenuating boundary-layer solutions.

It was shown in Refs 1–4 that the spectrum of homogeneous solutions for laminated bodies with alternating hard and soft layers can be divided into "lower" and "higher" parts, where a certain applied theory always corresponds to the lower part, which takes into account a weakly attenuating boundary-layer solution. The method was extended in Ref. 5 to problems of steady torsional oscillations of a radially laminated cylinder with alternating hard and soft layers. The spectrum of the homogeneous solutions for the vartex problem of a triple-layered spherical shell with a soft filler was investigated in Ref. 6.

1. Formulation of the boundary-value problem for a radially non-uniform spherical layer

We will first consider the problem of the torsion of a radially non-uniform spherical layer. We will denote by $V=\{r\in[r_0, r_1], \theta\in[\theta_0, \theta_1], \varphi\in[0, 2\pi]\}$ the region occupied by the shell (r, θ, φ) are spherical coordinates). We will assume that the shear modulus $G=G(r)$ is an arbitrary strictly positive integrable function.

The equilibrium equation in a spherical system of coordinates has the form[7]

$$Lu_\varphi \equiv L_r u_\varphi + L_\theta u_\varphi = 0$$

$$L_r u_\varphi = \frac{1}{rG}\frac{\partial}{\partial r}\left[r^4 G\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right], \quad L_\theta u_\varphi = \left(\frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} + 2\right)u_\varphi \tag{1.1}$$

where u_φ is a component of the displacement vector.

We will also assume that the faces are stress-free

$$\sigma_{r\varphi} = l_r u_\varphi = Gr\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = 0 \text{ when } r = r_\alpha, \quad \alpha = 0, 1 \tag{1.2}$$

while on the conical surfaces (ends)

$$\sigma_{\theta\varphi} = \frac{G}{r}\left(\frac{\partial u_\varphi}{\partial\theta} - \operatorname{ctg}\theta\, u_\varphi\right) = \tau_\alpha \text{ when } \theta = \theta_\alpha \tag{1.3}$$

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The solution will be sought in the form

$$u_\varphi = \upsilon(r)m(\theta) \tag{1.4}$$

where m(θ) is the solution of the Legendre equation[8]

$$m'' + \operatorname{ctg}\theta m' + (z^2 - 1/4 - \csc^2\theta)m = 0 \tag{1.5}$$

After substituting (1.4) into relations (1.1) and (1.2) we obtain the eigenvalue problem

$$A\upsilon = \lambda\upsilon$$
$$A\upsilon = \{-L_r\upsilon, l_r\upsilon|_{r=r_\alpha} = 0\}, \quad \lambda = 9/4 - z^2 \tag{1.6}$$

We will introduce a Hilbert space H with scalar product

$$(u, w)_H = \int_{r_0}^{r_1} Guwdr$$

Theorem 1. *The operator A: H → H is non-negative.*

Proof. Consider the quadratic form of the operator A. After integration by parts and taking the conditions $l_r v|_{r=r_\alpha} = 0$ into account, we obtain

$$(A\upsilon, \upsilon)_H = \int_{r_0}^{r_1} G\left(\frac{d\upsilon}{dr} - \frac{\upsilon}{r}\right)^2 r^2 dr \ge 0$$

The proof of the theorem follows from this inequality.

Corollary 1. *The non-zero eigenvalues of the operator A $\lambda_s > 0$, $\lambda_s \to \infty$ as $S \to \infty$.*

Corollary 2. *The set of eigenfunctions $\{v_s\}_{S=0}^{\infty}$ forms an orthogonal basis of the space H,*[9] *i.e.*

$$(\upsilon_s, \upsilon_t)_H = d_s\delta_{st}, \quad d_s = (\upsilon_s, \upsilon_s)_H = \int_{r_0}^{r_1} G\upsilon_s^2 dr \tag{1.7}$$

and for all $\upsilon \in H$ can be represented in the form

$$\upsilon = \sum_{s=0}^{\infty} c_s\upsilon_s, \quad c_s = \frac{(\upsilon, \upsilon_s)_H}{d_s}$$

where $\lambda = \lambda_0 = 0$, the eigenvalue of the operator A: H → H, and the eigenfunction $v_0(r) = r$ corresponds to it.

The expression

$$u_{\varphi s} = m_s(\theta)\upsilon_s(r) \tag{1.8}$$

will be called the elementary solution. Note that

$$m_0 = A_0\sin\theta + \tilde{B}_0(\theta), \quad \tilde{B}_0(\theta) = B_0\left(\frac{1}{2}\sin\theta\ln\left(\operatorname{ctg}^2\frac{\theta}{2}\right) + \operatorname{ctg}\theta\right)$$
$$m_s = A_sP^1_{z_s-1/2}(\cos\theta) + B_sQ^1_{z_s-1/2}(\cos\theta), \quad z_s = \sqrt{9/4 - \lambda_s} \tag{1.9}$$

where $P^1_{z_s-1/2}(\cos\theta)$, $Q^1_{z_s-1/2}(\cos\theta)$ are associated Legendre functions of the first and second kind, respectively, and A_s and B_s are arbitrary constants.

The above analysis enables any solution of Eq. (1.1), which satisfies boundary conditions (1.2), to be represented in the form

$$u_\varphi = u_{\varphi 0} + \sum_{s=1}^{\infty} u_{\varphi s} \tag{1.10}$$

Then the arbitrary constants in expressions (1.9) are defined when the boundary conditions are satisfied on the ends.

As an example, consider the problem with boundary conditions (1.3). From (1.8)-(1.10) we have

$$\sigma_{\theta\varphi} = -\frac{2B_0G}{\sin^2\theta} + \frac{G}{r}\sum_{s=1}^{\infty} f_s(\theta)\upsilon_s(r); \quad f_s(\theta) = m_s' - m_s\operatorname{ctg}\theta \tag{1.11}$$

Substituting (1.11) into condition (1.3) and multiplying scalarly by v_t ($t = 0, 1, \ldots$), and taking conditions (1.7) into account, we obtain

$$B_0 = -\frac{\sin^2\theta_0}{2d_0}\int_{r_0}^{r_1}\tau_0 r^2 dr = -\frac{\sin^2\theta_1}{2d_0}\int_{r_0}^{r_1}\tau_1 r^2 dr \tag{1.12}$$

$$f_s(\theta_0) = b_{0s}, \quad f_s(\theta_1) = b_{1s} \tag{1.13}$$

where

$$b_{\alpha s} = \frac{1}{d_s}\int_{r_0}^{r_1}\tau_\alpha v_s r dr, \quad d_0 = \int_{r_0}^{r_1} G r^2 dr$$

Simultaneous satisfaction of the two equalities (1.12), defining B_0, is equivalent to satisfying the equilibrium condition of the shell. Note that the torque in section θ = const is defined by the expression

$$M = 2\pi\sin^2\theta\int_{r_0}^{r_1}\sigma_{\theta\varphi} r^2 dr = -4\pi B_0 d_0 \tag{1.14}$$

The constant $\psi = -B_0$ can be regarded as the relative torsion angle, while $D = 4\pi d_0$ can be regarded as the torsional rigidity.

2. Analysis of the eigenvalue problem for a radially laminated sphere

We will now analyse eigenvalue problem (1.16) for a laminated shell with alternating hard and soft layers. Suppose $n = 2l - 1$ is the total number of layers. We will assume that the inner and outer layers are hard. Each hard layer is given an odd number $j = 1, 3, \ldots, n$ and each soft layer is given an even number $i = 2, 4, \ldots, n-1$. We will assume, for simplicity, that the elastic properties in all the hard and soft layers are the same: the shear moduli $G_j = G_g$ and $G_i = G_m$. The inner radius of the k-th layer will be denoted by r_{0k} and the outer radius will be denoted by r_{1k} ($k = 1, 2, \ldots, n$).

We will introduce the following notation: $v = v_k$ if $r \in [r_{0k}, r_{1k}]$. If the individual layers are rigidly connected (without slippage), problem (1.6) reduces to the following interface problem in v_k

$$\begin{gathered} v_k'' + 2r^{-1}v_k' - r^{-2}(z^2 - 1/4)v_k = 0 \\ G_1(v_1'(r) - r^{-1}v_1(r))\big|_{r_{01}} = G_n(v_n'(r) - r^{-1}v_n(r))\big|_{r_{1n}} = 0 \end{gathered} \tag{2.1}$$

$$\begin{gathered} G_1(v_t'(r) - r^{-1}v_t(r))\big|_{r_{1t}} = G_{t+1}(v_{t+1}'(r) - r^{-1}v_{t+1}(r))\big|_{r_{0,t+1}} \\ v_t(r)\big|_{r_{1t}} = v_{t+1}(r)\big|_{r_{0,t+1}} \\ k = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, n-1 \end{gathered} \tag{2.2}$$

We will introduce the small parameter $p = G_m/G_g$ as a characteristic of the relative stiffness of the layers and we will formulate the fundamental results of an investigation of eigenvalue problem (2.1), (2.2) as $p \to 0$.

Theorem 2. *The spectrum $\Lambda(p)$ of problem (2.1), (2.2) can be regarded as a combination of three sets*

$$\Lambda(p) = \Lambda_0(p) \cup \Lambda_1(p) \cup \Lambda_2(p)$$

Here

1) $\Lambda_0(p)$ consists of the eigenvalue $z_0^\pm = \pm 3/2$;
2) $\Lambda_1(p)$ consists of $2(l-1)$ real eigenvalues of the form

$$z_t^\pm = \pm[3/2 - p\gamma_t + O(p^2)] \tag{2.3}$$

where γ_t are non-zero eigenvalues of the homogeneous algebraic system

$$C\mathbf{X} - \gamma B\mathbf{X} = \mathbf{0} \tag{2.4}$$

$$C = \left\| \begin{array}{ccccc} c_{11} & -c_{11} & 0\ldots & 0 & 0 \\ -c_{11} & c_{11}+c_{33} & -c_{33}\ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\ldots & -c_{n-2,n-2} & c_{n-2,n-2} \end{array} \right\|$$

$$c_{jj} = \frac{r_{1j}^3 r_{0,j+2}^3}{r_{0,j+2}^3 - r_{1j}^3}, \quad j = 1, 3, \ldots, n-2, \quad \mathbf{X} = (X_1, X_3, \ldots, X_n)^T$$

$$B = \operatorname{diag}\|b_{jj}\|, \quad b_{jj} = \frac{r_{1j}^3 - r_{0j}^3}{3}, \quad j = 1, 3, \ldots, n$$

3) $\Lambda_2(p)$ consists of $2n$ denumerable sets of eigenvalues of the form

$$z_{kq}^{\pm} = \pm i\beta_{kq} + O(p^{\delta}), \quad k = 1, 2, \ldots, n, \quad q = 1, 2, \ldots \tag{2.5}$$

where

$$\beta_{kq} = \frac{\pi q}{\varepsilon\omega_k}, \quad \omega_k = \frac{1}{\varepsilon}\ln\frac{r_{1k}}{r_{0k}}, \quad \varepsilon = \ln\frac{r_{1n}}{r_{01}}$$

$$\delta = \begin{cases} 1, & \text{if} \quad z_{k+1,q} \neq z_{k,q}, \; z_{kq} \neq z_{k-1,q} \\ 1/2, & \text{if} \quad z_{k+1,q} = z_{k,q} \;\text{ or }\; z_{k,q} = z_{k-1,q} \end{cases}$$

and ε is a parameter characterizing the thin-walled form of the shell.

Note that eigenvalues (2.5) with odd k correspond to the hard layers while those with even k correspond to the soft layers. We will denote the set of these eigenvalues by $\Lambda_{2,j}(p)$, $\Lambda_{2,i}(p)$ respectively.

Since the quadratic forms

$$(C\mathbf{X}, \mathbf{X}) = \sum_{j=1,3,\ldots}^{n-2} c_{jj}(X_j - X_{j+2})^2 \geq 0, \quad (B\mathbf{X}, \mathbf{X}) = \sum_{j=1,3,\ldots}^{n} b_{jj}X_j^2 > 0$$

then $\gamma_0 = 0$ is an eigenvalue, to which the eigenvector $\mathbf{X}_0 = (X_0, \ldots, X_0)^T$ corresponds, while for $t = 1, \ldots, l\text{-}1$ we have

$$\gamma_t = (C\mathbf{X}_t, \mathbf{X}_t)/(B\mathbf{X}_t, \mathbf{X}_t) > 0$$

which corresponds to Corollary 1 ($\lambda_t = 9/4 - z_t^2 \approx 3p\gamma_t > 0$).

If the relative thickness of the shell is small, it follows from an analysis of system (2.4) that

$$\gamma_t = \varepsilon^{-1}\gamma_t^{(0)} + O(\varepsilon^0); \quad \gamma_t^{(0)} > 0 \tag{2.6}$$

We will only give a general outline of the proof of the Theorem 2.

Eigenvalue problem (2.1), (2.2) reduces to investigating a homogeneous algebraic system with a matrix, whose elements depend analytically on the eigenvalue parameter z and linearly on the parameter p. The results formulated above were obtained by applying perturbation theory with respect to the parameter p (Ref. 10) to this algebraic system. Here it is important to analyse the limiting problem. In this case when p = 0 we have two limiting cases: 1) $E_m \to 0$ and the shear modulus G_g is finite, and 2) $G_g \to \infty$ and the shear modulus G_m is finite. A system of spherical shells (hard layers) connected to one another, the spherical surfaces of which are stress-free, corresponds to the first limiting case. The spectrum of limit problem will be a combination of sets of eigenvalues of the eigenvalue problems corresponding to the individual homogeneous spherical shells. We will denote these sets by $\Lambda_j(0)$. Each of these consists of $z_{0j}^{\pm} = \pm 3/2$ and roots of the equation

$$\operatorname{sh}(z\varepsilon\omega_j) = 0 \tag{2.7}$$

For small $p \neq 0$ the numbers $z_{0j}^{\pm}$ generate the set $\Lambda_1(p)$. A system of l-1 spherical shells (soft layers), unconnected with one another, on whose spherical surfaces the displacements are zero, correspond to the second limiting case. The spectrum of the limit problem will be the sum of l-1 sets of eigenvalues. Each such set $\Lambda_i(0)$ consists of the roots of the equation

$$\operatorname{sh}(z\varepsilon\omega_i) = 0 \tag{2.8}$$

The roots of Eqs. (2.7) and (2.8) generate the set $\Lambda_2(p)$.

The following thorough solution corresponds to the eigenvalue $z_0^{\pm} = \pm 3/2$

$$u_{\varphi k}^{(0)}(r, \theta) = r(A_0\sin\theta + \tilde{B}_0(\theta)) \tag{2.9}$$

Eigenvalues v_k, corresponding to the eigenvalue of $\Lambda_1(p)$, have the following form

$$v_{kt}(r) = v_{kt0}(r) + O(p) \tag{2.10}$$

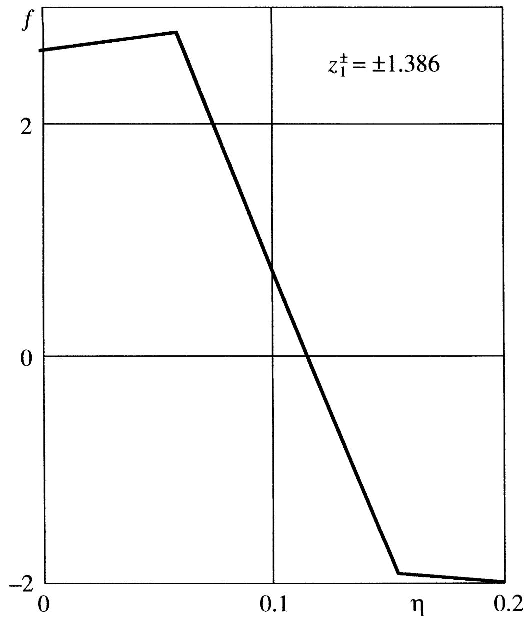

Fig. 1.

$$\upsilon_{jt0} = rX_{tj}$$
$$\upsilon_{it0} = \frac{1}{r^2(r_{1i}^3 - r_{0i}^3)}[r_{1,i-1}r_{0i}^2(r_{1i}^3 - r^3)X_{t,i-1} + r_{1i}^2 r_{0,i+1}(r^3 - r_{0i}^3)X_{t,i+1}] \tag{2.11}$$

The specific form of the eigenfunction, corresponding to the set Λ_2(p), depends mainly on two factors: first, to which of the sets $\Lambda_{2,j}(p)$ or $\Lambda_{2,i}(p)$ the eigenvalue belongs, and second, whether among the limit values of the eigenvalues there are multiple ones or not. For example, if $\beta_{iq} \in \Lambda_{2,i}(0)$ and there is a simple eigenvalue, the eigenfunction is localized in the corresponding soft layer. If $\beta_{jq} \in \Lambda_{2,j}(0)$ and there is a simple eigenvalue, the eigenfunction is localized in the corresponding hard layer and the soft layer surrounding it. We will call this feature of the distribution of the eigenfunctions over the shell thickness "localization of the eigenforms". In the case of multiple limit eigenvalues, the distribution pattern of the eigenfunctions becomes more complex. As follows from (2.5), such a situation arises when, for two or several layers, the ratios $\lambda_{qs} = q/\omega_s$ become equal for different s. Note that for periodic structures λ_{qs} are the same for hard and for soft layers, and hence for such structures all the eigenforms corresponding to the set $\Lambda_2(p)$, are spread over the thickness.

We will illustrate the properties of the eigenfunctions by graphs obtained from the exact solution of the problem for a triplelayered shell with the following values of the dimensionless parameters

$$p = 0.001, \quad \omega_1 = 0.273, \quad \omega_2 = 0.497, \quad \omega_3 = 0.230$$

On the graphs $\eta = \varepsilon^{-1} \ln(r/r_{01}) \in [0, 1]$ is the dimensionless radial coordinate.

In the case considered, the set $\Lambda_1(p)$ consists of only two eigenvalues $z_1^{\pm} = \pm 1.386$ and the eigenfunction f(η), shown in Fig. 1, corresponds to them. The graphs in Fig. 2 illustrate the location of the eigenforms. The eigenfunctions corresponding to the eigenvalues $z_{21} = 3.142i$ and $z_{22} = 6.283i$ are shown on the left of Fig. 2. These two forms are localized in the middle layer. The first part of Fig. 2 illustrates the localization of the eigenfunction in the soft and inner hard layer (eigenvalues $z_{11} = 3.140i$ and $z_{12} = 6.283i$). Note that the eigevalues $z_{11}$and z_{21} differ solely in the fourth decimal place, but the eigenforms corresponding to them are essentially different.

When describing the properties of the distribution of the eigenfunctions one must add the word "as a rule", since our analysis shows that deviations from the rules formulated are possible. Moreover, the distribution of the amplitude of the stresses $\sigma_{r\varphi}$ over the thickness by no means repeats the distributions of the amplitudes of the displacements of the corresponding elementary solutions.

3. Analysis of the elementary solutions

We will now analyse the behaviour of the elementary solutions corresponding to the sets $\Lambda_1(p)$ and $\Lambda_2(p)$, which reduces to investigating the behaviour of the solution of the equations

$$m_s'' + \operatorname{ctg}\theta m_s' + (z_s^2 - 1/4 - \csc^2\theta)m_s = 0 \tag{3.1}$$

in the neighbourhood of the ends $\theta = \theta_\alpha$. We will assume that $\theta_0 \neq 0$and $\theta_1 \neq \pi$.

If $z_s \in \Lambda_1(p)$, it can be shown, using the asymptotic formulae (2.3) and (2.6), that the principal term of the asymptotic solution has the form

$$m_s = \begin{cases} \exp[\mu_{s0}^{-}(\theta - \theta_0)] & \text{for} \quad \theta = \theta_0 \\ \exp[\mu_{s1}^{+}(\theta - \theta_1)] & \text{for} \quad \theta = \theta_1 \end{cases} \tag{3.2}$$

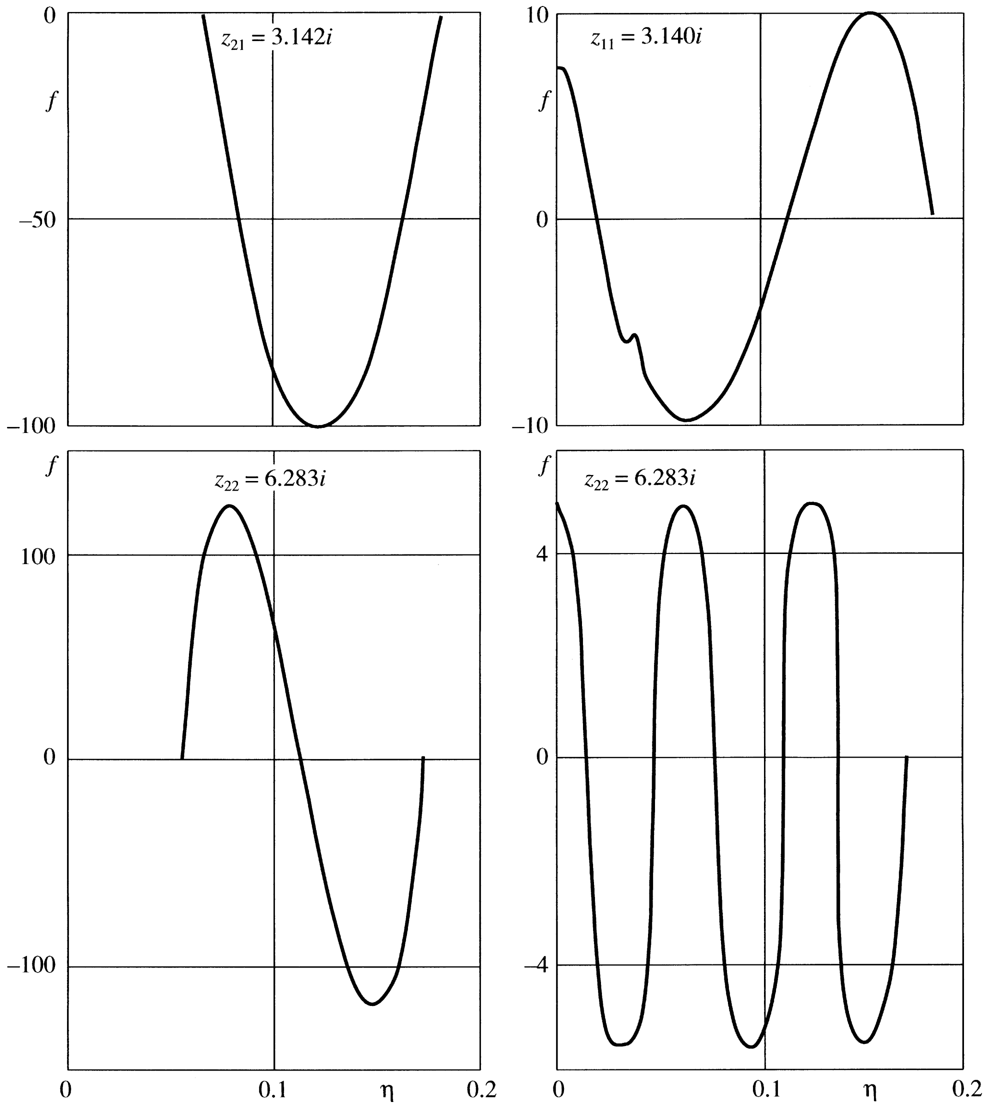

Fig. 2.

where

$$2\mu^{\pm}_{s\alpha} = -\operatorname{ctg}\theta_\alpha \pm \sqrt{\operatorname{ctg}^2\theta_\alpha + 4a_{s\alpha}}, \quad a_{s\alpha} = 3\varepsilon^{-1}p\gamma_s^{(0)} + \sin^{-2}\theta_\alpha\cos 2\theta_\alpha$$

If follows from formula (3.2) that: 1) the elementary solutions corresponding to given eigenvalues attenuate exponentially with distance from the ends, and 2) the rate of attenuation is determined by the position of the conic section and the ratio p/ε.

If $z_s = z_{q,k} \in \Lambda_2(p)$, the principal term of the asymptotic solution of Eq. (3.1) has the form

$$m_s = m_{q,k} = \begin{cases} \exp(-\beta_{kq}(\theta-\theta_0)) & \text{for } \theta = \theta_0 \\ \exp(\beta_{kq}(\theta-\theta_1)) & \text{for } \theta = \theta_1 \end{cases} \tag{3.3}$$

The set of elementary solutions, corresponding to the set $\Lambda_1(p)$ (the set $\Lambda_2(p)$), will be called a weak (strong) boundary layer.

Hence, as in the case of a laminated plate[1] and a radially laminated cylinder,[2–4] the complete solution (1.10) for a thin laminated spherical shell with alternating hard and soft layers consists of the "thorough solution" $u_{\varphi 0}$, which is determined by the torsional moment M, the shear stresses $\tau_0$and τ_1, applied to the boundary conical surfaces, the weak boundary layer and the strong boundary layer, which are determined by the self-balanced part of the external load. In this case the weak boundary layer may have a considerable effect on the internal stress–strain state of the shell, which indicates that the Saint-Venant principle in classical formulation[11] is violated.

4. Construction of the applied theory of the torsion of a radially laminated sphere

It was shown in Refs 1-5, 11 that the lower part of the spectrum corresponds to a certain applied theory, which includes all the boundary layers together with the thorough solutions the thorough solutions. The thorough solutions with a weak boundary layer can be given the following mechanical interpretation.

We will assume that the displacements of the points of the hard layer have the form

$$u_{\varphi j}(r,\theta) = rg_j(\theta) \tag{4.1}$$

We will use the fact that the displacements in the soft layer in any section θ = const are determined by the displacements of the adjacent hard layers, i.e.,

$$u_{\varphi i}(r_{0i},\theta) = r_{0i}g_{i-1}(\theta),\quad u_{\varphi i}(r_{1i},\theta) = r_{1i}g_{i+1}(\theta)$$

This hypothesis enables the displacements in the soft layer to be represented in the form

$$u_{\varphi i}(r,\theta) = \frac{1}{r^2(r_{1i}^3-r_{0i}^3)}[r_{0i}^2r_{1,i-1}(r_{1i}^3-r^3)g_{i-1}(\theta)+r_{1i}^2r_{0,i+1}(r^3-r_{0i}^3)g_{i+1}(\theta)] \tag{4.2}$$

According to relations (4.1) and (4.2) the stress–strain state in each hard and soft layer will be as follows:

$$\sigma_{\theta\varphi}^{(j)} = 2G_g\varepsilon_{\theta\varphi}^{(j)},\quad \varepsilon_{\theta\varphi}^{(j)} = \frac{1}{2}(g_j'(\theta)-g_j(\theta)\operatorname{ctg}\theta)$$

$$\sigma_{r\varphi}^{(i)} = 2G_m\varepsilon_{r\varphi}^{(i)},\quad \varepsilon_{r\varphi}^{(i)} = \frac{3r_{0i}^3r_{1i}^3}{2r^3(r_{1i}^3-r_{0i}^3)}(g_{i+1}(\theta)-g_{i-1}(\theta)) \tag{4.3}$$

The remaining components of the stress and strain tensors are equal to zero.

In order to obtain the boundary-value problem corresponding to the chosen stress–strain model, we will use the Legrange variational principle[7]

$$\delta\Pi-\delta A = 0 \tag{4.4}$$

where δΠ is the variation of the strain energy and δA is the variation of the work of the external forces. By equalities (4.3) we have

$$\Pi = \pi\int_0^{\pi}\left[\sum_{j=1,3,\ldots}^{n}\int_{r_{0j}}^{r_{1j}}\sigma_{\theta\varphi}^{(j)}\varepsilon_{\theta\varphi}^{(j)}r^2dr+\sum_{j=2,4,\ldots}^{n}\int_{r_{0i}}^{r_{1i}}\sigma_{r\varphi}^{(i)}\varepsilon_{r\varphi}^{(i)}r^2dr\right]\sin\theta d\theta \tag{4.5}$$

To determine δA we will use the fact that the following boundary conditions are specified on the conic sections

$$\sigma_{\theta\varphi}^{(k)}(r,\theta_0) = h_k(r),\quad u_{\varphi k}(r,\theta_1) = 0 \tag{4.6}$$

where

$$h_k(r) = \begin{cases} h_j(r), & r\in[r_{0j};r_{1j}] \\ 0, & r\in[r_{0i};r_{1i}]\end{cases};\quad k = 1,2,\ldots,n$$

Assuming that δg_j are independent variations ($\delta g_j = 0$ when $\theta = \theta_1$), taking relations (4.1)-(4.3) and (4.5) and (4.6) into account, we obtain the following boundary-value problem from (4.4)

$$B\mathbf{g}''+\operatorname{ctg}\theta\cdot B\mathbf{g}'+(1-\operatorname{ctg}^2\theta)B\mathbf{g}+3pC\mathbf{g} = \mathbf{0} \tag{4.7}$$

$$(B\mathbf{g}'-\operatorname{ctg}\theta\cdot B\mathbf{g})\sin\theta|_{\theta=\theta_0} = \mathbf{F},\quad \mathbf{g}|_{\theta=\theta_1} = \mathbf{0} \tag{4.8}$$

where

$$\mathbf{g} = (g_1,g_3,\ldots,g_n)^T,\quad \mathbf{F} = (F_1,F_3,\ldots,F_n)^T,\quad F_j = \frac{2}{G_j}\int_{r_{0j}}^{r_{1j}}r^2h_j(r)dr$$

We will put

$$\mathbf{g}(\theta) = \mathbf{X}m(\theta)$$

Substituting this expression into Eq. (4.7), we have

$$(9/4-z^2)B\mathbf{X}+3pC\mathbf{X} = \mathbf{0} \tag{4.9}$$

If we make the replacement z^2 - 9/4 = 3pγ here, we obtain algebraic system (2.4). Hence $\gamma_0 = 0\,(z_0^{\pm} = \pm 3/2)$ is an eigenvalue and the eigenfunction $\mathbf{X}_0 = (X_0,\ldots,X_0)^T$ corresponds to it. The eigfunction $\mathbf{X}_t = (X_{t1},X_{t3},\ldots,X_{tn})$ of system (4.9) satisfies the orthonormalization condition

$$(B\mathbf{X}_t,\mathbf{X}_k) = \sum_{j=1,3,\ldots}^{n}b_{jj}X_{tj}X_{kj} = \delta_{tk} \tag{4.10}$$

Note that $z_0^{\pm} = \pm 3/2$ are eigenvalues of Eq. (4.9) and the corresponding eigenfunction has the form

$$\mathbf{X}_0 = (X_0, \ldots, X_0)^T; \quad X_0 = \left(\sum_{j=1,3,\ldots}^{n} b_{jj}\right)^{-1/2}$$

The general solution of vector Eq. (4.7) can be represented in the form

$$\mathbf{g} = \mathbf{g}_0 + \sum_{t=1}^{l-1} X_t(A_t P^1_{z_t - 1/2}(\cos\theta) + B_t Q^1_{z_t - 1/2}(\cos\theta)) \tag{4.11}$$

$$\mathbf{g}_0 = \mathbf{X}_0(A_0 \sin\theta + \tilde{B}_0(\theta)) \tag{4.12}$$

By constructing the displacement field corresponding to the vector $\mathbf{g}_0$, we obtain from (4.1), (4.2) and (4.12)

$$u^{(0)}_{\varphi k} = r X_0(A_0 \sin\theta + \tilde{B}_0(\theta)), \quad k = 1, 2, \ldots, n$$

i.e., this particular solution is completely equivalent to the thorough solution.

The distribution of the displacements over the radius, corresponding to $z_t \neq \pm 3/2$, is given by (2.11), in complete agreement with hypotheses (4.1) and (4.2).

We determine the constants A_0, B_0, A_t and B_t from boundary conditions (4.8). Substituting expression (4.11) into conditions (4.8) and using orthagonality condition (4.10), we obtain

$$B_0 = -\frac{X_0 \sin\theta_0}{2} \sum_{j=1,3,\ldots}^{n} F_j, \quad A_0 = -\frac{\tilde{B}_0(\theta)}{\sin\theta_1}$$

$$A_t = -Q^1_{z_t - 1/2}(\cos\theta_1)\chi_t, \quad B_t = P^1_{z_t - 1/2}(\cos\theta_1)\chi_t$$

Here

$$\chi_t = \frac{1}{\Delta_t} \sum_{j=1,3,\ldots}^{n} F_j X_{tj}$$

$$\Delta_t = 2\cos\theta_0 D^{(1,1)}_{z_t - 1/2}(\theta_0, \theta_1) + \left(z_t^2 - \frac{1}{4}\right)\sin\theta_0 D^{(0;1)}_{z_t - 1/2}(\theta_0, \theta_1)$$

$$D^{(s;q)}_{z_t - 1/2}(\theta_0, \theta_1) = P^s_{z_t - 1/2}(\cos\theta_0) Q^q_{z_t - 1/2}(\cos\theta_1) - P^q_{z_t - 1/2}(\cos\theta_1) Q^s_{z_t - 1/2}(\cos\theta_0)$$

$$s = 0, 1; \quad q = 0, 1$$

As follows from relations (4.3), in the approximate model considered here the soft layers play the role of a Winkler foundation, operating under shear.

The results described, and also the results of investigations for laminated plates with alternating hard and soft layers,[12] indicate that the most adequate applied theory for similar structural components is a theory based on the following hypotheses: 1) the Kirchhoff–Love hypothesis for rigid layers, and 2) the Winkler hypothesis on shear and transverse compression for soft layers. Moreover, within the framework of this theory for certain boundary conditions one can obtain the stress–strain state corresponding to the hypothesis of a common normal for the whole packet.

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